From Syllogism to Common Sense

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Normal Modal Logic

Kripke Semantics
Completeness
and Correspondence Theory

Lecture 9

Examples of Modal Logics

Classic Distinctions between Modalities

- **Alethic modality:** necessity, possibility, contingency, impossibility
  - distinguish further: *logical* - *physical* - *metaphysical*, etc.
- **Temporal modality:** always, some time, never
- **Deontic modality:** obligatory, permissible
- **Epistemic modality:** it is known that
- **Doxastic modality:** it is believed that

Technically, all these modalities are treated
in the same way, by using unary modal operators

Examples of Modal Logics

Modern interpretations of modalities

- **Mathematical Logic:**
  - The logic of proofs **GL**: [] A means: In PA it is provable that 'A'.
- **Computer Science:**
  - Linear Temporal Logic LTL: Formal Verification
    - **X** A : in the next moment 'A'
    - A **U** B: A is true until B becomes true
    - **G** = 'always' , **F** = 'eventually',
    - **liveness properties** state that something good keeps happening:
      - **G F** A or also **G (B -> F** A)
- **Linguistics / KR / etc.**

Modal Logic: Some History

- Modern modal logic typically begins with the systems devised by C. I. Lewis, intended to model **strict implication** and avoid the paradoxes of material implication, such as the '***ex falso quodlibet***'.
  - " If it never rains in Copenhagen, then Elvis never died."
  - (No variables are shared in example => relevant implication)
- For strict implication, we *define* **A ~~> B** by **[] (A --> B)**
- These systems are however mutually incompatible, and no **base logic** was given of which the other logics are extensions of.
- The modal logic **K** is such a base logic, named after Saul Kripke, and which serves as a minimal logic for the class of all its (**normal) extensions** - defined next via a Hilbert system.

A Hilbert system for Modal Logic K

- The following is the ***standard Hilbert system*** for the modal logic **K**.

Axioms

$$p_1 \to (p_2 \to p_1)$$
$$(p_1 \to p_2) \to (p_1 \to (p_2 \to p_3)) \to (p_1 \to p_3)$$
$$p_1 \to p_1 \vee p_2$$
$$p_2 \to p_1 \vee p_2$$
$$(p_1 \to p_3) \to (p_2 \to p_3) \to (p_1 \vee p_2 \to p_3)$$
$$(p_1 \to p_2) \to (p_1 \to \neg p_2) \to \neg p_1$$
$$\neg\neg p_1 \to p_1$$
$$p_1 \wedge p_2 \to p_1$$
$$p_1 \wedge p_2 \to p_2$$
$$p_1 \to p_2 \to p_1 \wedge p_2$$
$$\Box(p \to q) \to (\Box p \to \Box q)$$

two classical tautologies instead of $\bot \to \mathbf{p}$ in INT

new axiom of **Box Distribution**

Rules

$$\frac{p_1 \qquad p_1 \to p_2}{p_2} \qquad \frac{p}{\Box p}$$

new rule of **Necessitation**

Some More Modal Frege systems

- **Hilbert systems** for other modal logics are obtained by adding axioms.

modal logic	axioms		
$K4$	K	$+$	$\Box p \to \Box\Box p$
KB	K	$+$	$p \to \Box\Diamond p$
GL	K	$+$	$\Box(\Box p \to p) \to \Box p$
$S4$	$K4$	$+$	$\Box p \to p$
$S4Grz$	$S4$	$+$	$\Box(\Box(p \to \Box p) \to p) \to \Box p$

- More generally, in a fixed language, the class of all **normal modal logics** is defined as any set of formulae that
  - (1) contains K (2) is closed under **substitution** and (3) **Modus Ponens**
- In particular, any normal extension of K contains the **Axiom of Box-Distribution:**

$$\Box(p \to q) \to (\Box p \to \Box q)$$

Kripke Semantics

- A Kripke frame consists of a set **W**, the set of \`possible worlds', and a binary relation **R** between worlds. A valuation β assigns propositional variables to worlds. A **pointed model** $\mathbf{M_x}$ is a frame, together with a valuation and a distinguished world **x**.

$$
\begin{aligned}
M_x \models p \wedge q &\iff M_x \models p \text{ and } M_x \models q \\
M_x \models p \vee q &\iff M_x \models p \text{ or } M_x \models q \\
M_x \models p \to q &\iff \text{if } M_x \models p \text{ then } M_x \models q \\
M_x \models \neg p &\iff M_x \not\models p \\
M_x \models \Box p &\iff \text{for all } xRy : M_y \models p \\
M_x \models \Diamond p &\iff \text{exists } xRy : M_y \models p
\end{aligned}
$$

Modal Sat / Taut / Validity

- **Modal Sat:** A modal formula is satisfiable if there ***exists*** a pointed model that satisfies it.

- **Modal Taut:** A formula is a ***modal tautology*** if it is satisfied in ***all*** pointed models.

- **Modal Validity:** A formula is ***valid*** in a class of frames if it a modal tautology relative to that class of frames.

Check validity of Box Distribution

$$\Box(p \to q) \to (\Box p \to \Box q)$$

A Tableaux system for Modal Logic K

- Hilbert systems are generally considered difficult to **use** in a practical way.
- There are many proof systems for Modal Logics. One of the most popular ones are **Semantic Tableaux:**
  - refutation based proof system
  - highly developed optimisation techniques
  - allows to extract models directly from proofs
  - popular in particular for Description Logic based formalisms
  - often used for establishing upper bounds for the complexity of a SAT problem for a logic.

A Tableaux system for Modal Logic K

- In prefixed tableaux, every formula starts with a prefix and a sign
  - σ Z ϕ
- **Prefixes** (denoting possible worlds) keep track of accessibility.
  - A prefix σ is a finite sequence of natural numbers
  - Formulae in a tableaux are **labelled** with **T** or **F**.

Definition 1 (K prefix accessibility) *For modal logic* **K**, *prefix* σ' *is* accessible *from prefix* σ *if* σ' *is of the form* σn *for some natural number* n.

- Example 1 4 7 9 is accessible from 1 4 7 which is accessble from 1 4 etc.

A Tableaux system for Modal Logic K

- A basic semantic tableaux for K is given as follows:
- We introduce **prefixes** (denoting possible worlds) that keep track of accessibility.
- Formulae in the tableaux are **labelled** with T or F.
- We differentiate the following four kinds of formulas:

α	α_1	α_2
$TA \wedge B$	TA	TB
$FA \vee B$	FA	FB
$FA \rightarrow B$	TA	FB
$F\neg A$	TA	TA

Conjunctive

β	β_1	β_2
$TA \vee B$	TA	TB
$FA \wedge B$	FA	FB
$TA \rightarrow B$	FA	TB
$T\neg A$	FA	FA

Disjunctive

ν	ν_0
$T\Box A$	TA
$F\Diamond A$	FA

Universal

π	π_0
$T\Diamond A$	TA
$F\Box A$	FA

Existential

- These tables essentially encode the semantics of the logic.

A Tableau system for Modal Logic K

- A tableau is now expanded according to the following rules.
- A proof starts with assuming the falsity of a formula, and succeeds if every branch of the tableau closes, i.e. contains a direct contradiction.

Conjunctive	Disjunctive	Universal	Existential
$(\alpha)\quad \dfrac{\sigma\alpha}{\begin{array}{c}\sigma\alpha_1\\ \sigma\alpha_2\end{array}}$	$(\beta)\quad \dfrac{\sigma\beta}{\sigma\beta_1 \quad \sigma\beta_2}$	$(\nu^*)\quad \dfrac{\sigma\nu}{\sigma'\nu_0}$ [1]	$(\pi)\quad \dfrac{\sigma\pi}{\sigma'\pi_0}$ [2]

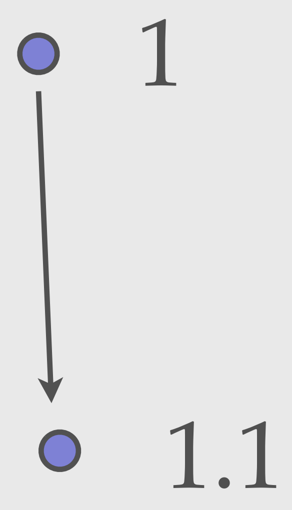
[1] σ' accessible from σ and σ' occurs on the branch already

[2] σ' is a simple unrestricted extension of σ, i.e., σ' is accessible from σ and no other prefix on the branch starts with σ'

A Tableau system for Modal Logic K

- We give an example derivation of a valid formula:

1	$F(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	(1)
1	$T\Box A \wedge \Box B$	(2) from 1
1	$F\Box(A \wedge B)$	(3) from 1
1	$T\Box A$	(4) from 2
1	$T\Box B$	(5) from 2
1.1	$FA \wedge B$	(6) from 3



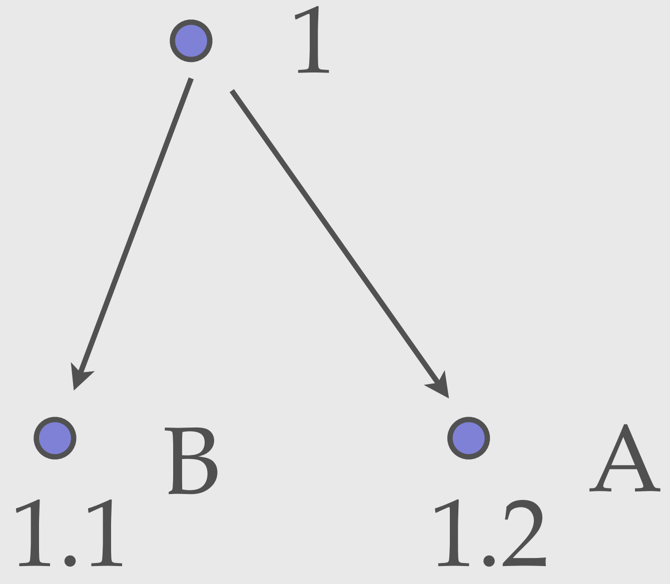
1.1	FA	(7) from 6	1.1	FB	(8) from 6
1.1	TA	(9) from 4	1.1	TB	(10) from 5
	$*$	7 and 9		$*$	10 and 8

- This shows **K**-validity of: $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$

A Tableau system for Modal Logic K

- We give a refutation of a satisfiable, but non-valid formula:

1	$F\Box(A \vee B) \rightarrow \Box A \vee \Box B$	(1)
1	$T\Box(A \vee B)$	(2) from 1
1	$F\Box A \vee \Box B$	(3) from 1
1	$F\Box A$	(4) from 3
1	$F\Box B$	(5) from 3
1.1	FA	(6) from 4
1.2	FB	(7) from 5
1.1	$TA \vee B$	(8) from 2
1.2	$TA \vee B$	(9) from 2



- This shows **K**-satisfiability of: $\Box(A \vee B) \wedge \Diamond\neg A \wedge \Diamond\neg B$

Kripke Semantics (Again)

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Completeness (Sketch)

- **Soundness:** Every **K**-provable formula is valid in all frames.
- **Completeness:** Every **K**-valid formula is **K**-provable.
  - **Lindenbaum Lemma:** Every consistent set of formulae can be extended to a maximally one.
  - **Canonical Models:** Construct worlds, valuations, and accessibility from the MCSs
  - **Truth Lemma:** Every consistent set is satisfied in the canonical model.

Canonical Models & Truth Lemma

- **Worlds** are maximally consistent sets MCSs
- **Valuations** are defined via membership in the MCSs
  - **Accessibility** is defined as follows

X **R** Y iff for every formula **A** we have

[] A ∈ X implies A ∈ Y

- or equivalently

X **R** Y iff for every formula **A** we have

<> A ∈ Y implies A ∈ X

Canonical Models & Truth Lemma

- **Worlds** are maximally consistent sets MCSs
- **Valuations** are defined via membership in the MCSs
  - **Accessibility** is defined as follows

X **R** Y iff for every formula **A** we have

<> A ∈ Y implies A ∈ X

- **Existence Lemma:** For any MCS w, if <> $\phi \in w$ then there is an accessible state v such that $\phi \in v$.

Note: this is the main difference to the classical completeness proof.

Canonical Models & Truth Lemma

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- **Valuations** are defined via membership in the MCSs
  - **Accessibility** is defined as follows

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- **Truth Lemma:** In the canonical model M we have

M, w ⊨ ϕ iff $\phi \in$ w.

Proof is almost immediate
from Existence Lemma and
the Definition of R

Characterising Modal Logics

- Most standard modal logics can be **characterised** via frame validity in certain classes of frames.
- A logic **L** is characterised by a class **F** of frames if **L** is **valid** in **F**, and any non-theorem $\phi \notin$ **L** can be **refuted** in a model based on a frame in **F**.

modal logic	characterising class of frames
K	all frames
K4	all transitive frames
KB	all symmetric frames
GL	R transitive, R^{-1} well-founded
S4	all reflexive and transitive frames
S4Grz	R reflexive and transitive, $R^{-1} - \mathsf{Id}$ well-founded

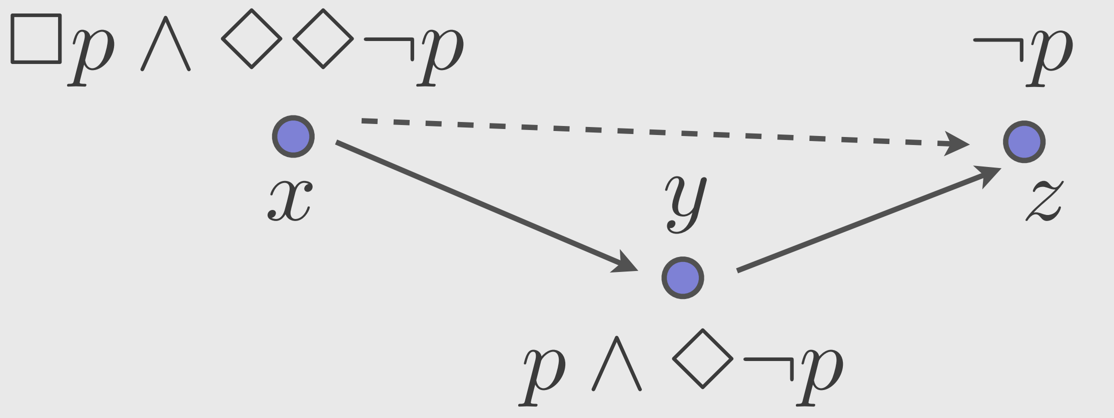
Correspondence Theory: Example

- We sketch as an example the correspondence between the modal logic axiom that defines the logic **K4** and the first-order axiom that characterises the **class of transitive frames**:

Let $\langle W, R \rangle$ be a frame. R is **transitive** if $\forall x, y, z \in W \ . \ xRy$ and yRz imply xRz

Theorem. $\Box p \to \Box\Box p$ is valid in a frame $\langle W, R \rangle$ iff R is transitive

- **Proof.**
  - (1) It is easy to see that the **4-axiom** is valid in transitive frames.
  - (2) Conversely, assume the **4-axiom** is refuted in a model $\mathbf{M}_x = <\mathbf{W},\mathbf{D}, \beta ,x>$



- The frame can clearly **not be transitive.**

Gödel–Tarski–McKinsey translation

- The Gödel–Tarski–McKinsey translation T, or simply **Gödel translation**, is an embedding of **IPC** into **S4**, or **Grz**.

$$
\begin{aligned}
\mathsf{T}(p) &= \Box p \\
\mathsf{T}(\bot) &= \bot \\
\mathsf{T}(\varphi \wedge \psi) &= \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \vee \psi) &= \mathsf{T}(\varphi) \vee \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \rightarrow \psi) &= \Box(\mathsf{T}(\varphi) \rightarrow \mathsf{T}(\psi))
\end{aligned}
$$

- Here, the Box Operator can be read as \`it is provable' or \`it is constructable'.

Gödel–Tarski–McKinsey translation

- **Theorem.** The Gödel translation is an embedding of **IPC** into **S4** and **Grz**.

$$\text{I.e. for every formula } \varphi \in \mathbf{IPC} \iff \mathsf{T}(\varphi) \in \mathbf{S4} \iff \mathsf{T}(\varphi) \in \mathbf{Grz}$$

- **Applications:**
  - modal companions of superintuitionistic logics

    $L \in \mathrm{NExt}(\mathbf{S4}) : \rho(L) = \{A \mid L \vdash \mathsf{T}(A)\}$

Rules: Admissible vs. Derivable

- The distinction between admissible and derivable rules was introduced by Paul Lorenzen in his 1955 book "Einführung in die operative Logik und Mathematik".
- Informally, a rule of inference **A/B** is **derivable** in a logic **L** if there is an **L -proof of B from A.**
- If there is an **L -proof of B from A**, by the rule of substitution there also is an **L -proof of σ(B) from σ(A)**, for any substitution **σ**. For admissible rules this has to be made explicit.
- A rule **A/B** is **admissible** in **L** if the set of theorems is closed under the rule, i.e. if for every substitution σ: **L** $\vdash$ **σ(A)** implies **L** $\vdash$ **σ(B)** . For this we usually write as:

$$A \mathrel{|\!\!\sim} B$$

Rules: Admissible vs. Derivable

- Therefore the addition of admissible rules leaves the set of theorems of a logic intact. Whilst they are therefore`redundant' in a sense, they can significantly shorten proofs, which is our main concern here.
- **Example:** Congruence rules.
- The general form of a rule is the following:

$$\frac{\phi_1, \ldots, \phi_n}{\phi}$$

- If our logic **L** has a 'well-behaved conjunction' (as in **CPC**, **IPC**, and most modal logics), we can always rewrite this rule by taking a conjunction and assume w.l.o.g. the following simpler form:

$$\frac{\psi}{\phi}$$

  - We are next going to show that in **CPC** (unlike many non-classical logics) the notions of admissible and derivable rule do indeed **coincide!**

CPC is Post complete

- A logic L is said to be **Post complete** if it has no proper consistent extension.
- **Theorem.** Classical PC is Post complete
- **Proof. (From** Chagrov & Zakharyaschev 1997**)**
  - Suppose **L** is a logic such that **CPC** $\subset$ **L** and pick some formula $\phi \in$ **L - CPC**.
  - Let **M** be a model refuting **ϕ.** Define a substitution **σ** by setting:

$$\sigma(p_i) := \begin{cases} \top & \text{if } M \models p_i \\ \bot & \text{otherwise} \end{cases}$$

  - Then **σ(ϕ)** does not depend on **M,** and is thus false in every model.
  - We therefore obtain **σ(ϕ)** $\rightarrow \bot \in$ **CPC.**
  - But since **σ(ϕ)** $\in$ **L,** we obtain $\bot \in$ **L** by MP, hence **L** is inconsistent. **QED**

CPC is 0-reducible

- A logic L is **0-reducible** if, for every formula $\phi \notin L$, there is a variable free substitution instance $\sigma(\phi) \notin L$.
- **Theorem.** Classical PC is 0-reducible.
- **Proof.**
  - Follows directly from the previous proof. **QED**
- **Note:** K is Post-incomplete and not 0-reducible.

CPC is Structurally Complete

- A logic **L** is said to be **structurally complete** if the sets of admissible and derivable rules coincide.
- **Theorem.** Classical PC is structurally complete.
- **Proof.**
  - It is clear that every derivable rule is admissible.
  - Conversely, suppose the rule: $\dfrac{\phi_1, \ldots, \phi_n}{\phi}$ is admissible in **CPC**, but not derivable.
  - This means that, by the Deduction Theorem $\phi_1 \wedge \ldots \wedge \phi_n \to \phi \notin CPC$
  - Since **CPC** is 0-reducible, there is a variable free substitution instance which is false in every model, i.e. we have $\sigma(\phi_1) \wedge \ldots \wedge \sigma(\phi_n) \to \sigma(\phi) \notin CPC$
  - This means that the formulae $\sigma(\phi_i)$ are all valid, while $\sigma(\phi)$ is not.
  - Therefore, we obtain: $\sigma(\phi_1) \wedge \ldots \wedge \sigma(\phi_n) \in CPC$
  - But $\sigma(\phi) \notin CPC$, which is a contradiction to admissibility. **QED**

Admissiblity in CPC is decidable

- **Corollary.** Admissibility in **CPC** is decidable.
- **Proof.** Pick a rule **A/B**. This rule is admissible if and only if it is derivable if and only if **A** → **B** is a tautology.
- Some Examples: **Congruence Rules:**

$$\frac{p \leftrightarrow q}{p \wedge r \leftrightarrow q \wedge r} \qquad \frac{p \leftrightarrow q}{p \vee r \leftrightarrow q \wedge r} \qquad \frac{p \leftrightarrow q}{p \to r \leftrightarrow q \to r}$$

$$\frac{p \leftrightarrow q}{r \wedge p \leftrightarrow r \wedge q} \qquad \frac{p \leftrightarrow q}{r \vee p \leftrightarrow r \wedge q} \qquad \frac{p \leftrightarrow q}{r \to p \leftrightarrow r \to q}$$

- if these are admissible in a logic **L** (they are derivable in **CPC**, **IPC**, **K**), the principle of **equivalent replacement** holds i.e.:

$$\psi \leftrightarrow \chi \in L \text{ implies } \phi(\psi) \leftrightarrow \phi(\chi) \in L$$

Admissible Rules in IPC and Modal K

- Intuitionistic logic as well as modal logics behave quite differently with respect to admissible vs. derivable rules (as well as many other meta-logical properties)
  - E.g., intuitionistic logic is not Post complete. Indeed there is a continuum of consistent extension of **IPC**, namely the class of **superintuitionistic logics;** the smallest Post-complete extension of **IPC** is **CPC**.
  - Unlike in **CPC**, the existence of admissible but not derivable rules is quite common in many well known non-classical logics, but there exist also examples of structurally complete modal logics, e.g. the Gödel-Dummett logic **LC**.
  - We next give some examples for **IPC** and modal **K**.
  - Finally, we will discuss how the sets of admissible rules can be presented in a finitary way, using the idea of a **base for admissible rules**.

Admissible Rules in Modal Logic

- The following rule is admissible, e.g., in the modal logics **K, D, K4, S4, GL.**
- It is **derivable** in **S4**, but it is not derivable in **K, D, K4,** or **GL.**

$$(\Box) \qquad \frac{\Box p}{p}$$

- **Proof.** (Derivability in **S4** and **K**):
- It is derivable in **S4** because $\Box p \to p$ is an axiom:
  - Assume a proof for $\Box p$ and apply MP once.

$$\Box p \to p$$

- It is not derivable in **K**: The formula $\Box^n p \to p$ is refuted in the one point irreflexive frame.

$\neg p$ • $\Box p$

- Note that the *classical Deduction Theorem* does not hold in modal logic!

Admissible Rules in Modal Logic

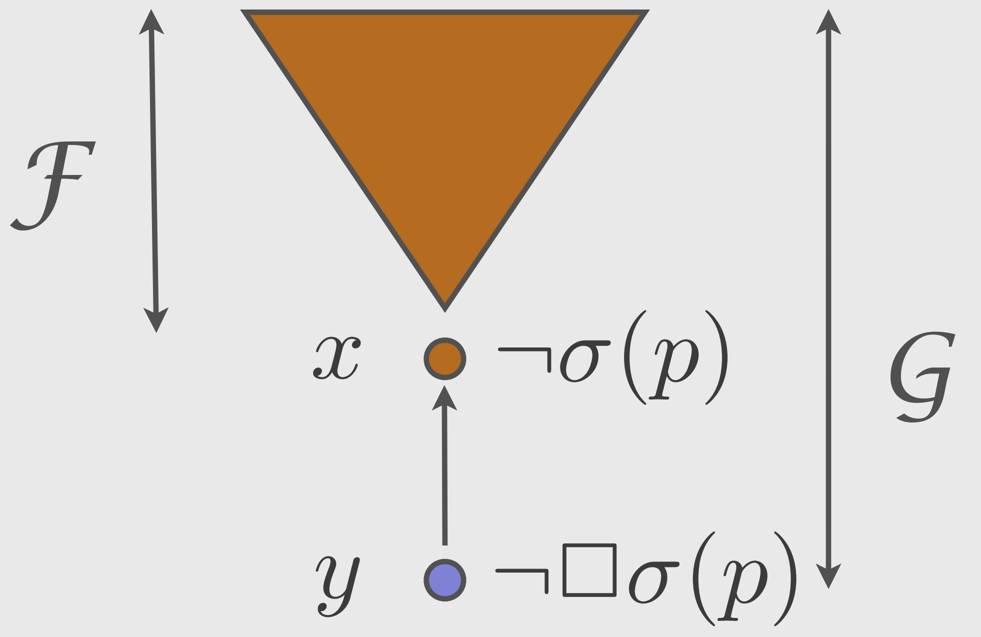
- The following rules is **admissible**, e.g., in the modal logics **K, D, K4, S4, GL.**
- It is derivable in **S4**, but it is not derivable in **K, D, K4,** or **GL.**

$$(\Box) \qquad \frac{\Box p}{p}$$

- **Proof.** (Admissibility in **K**):

Assume $\langle (F,R), \beta, x\rangle \not\models \sigma(p)$ for some frame (F,R). Pick some $y \notin F$, set $G = F \cup \{y\}$, $S = R \cup \{\langle y, x\rangle\}$, and $\gamma(p) = \beta(p)$ for all p. Then:

$\langle (G,S), \gamma, y\rangle \models \neg\Box\sigma(p)$ whilst we still have $\langle (G,S), \gamma, x\rangle \models \neg\sigma(p)$

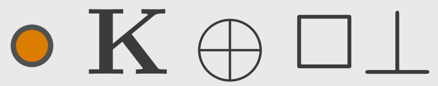


Admissible Rules in Modal Logic

- The following rules is admissible, e.g., in the modal logics **K, D, K4, S4, GL.**
- It is derivable in **S4**, but it is not derivable in **K, D, K4,** or **GL.**
- It is **not admissible** in some extensions of **K**, e.g.: $\mathbf{K} \oplus \Box\bot$

$$(\Box) \qquad \frac{\Box p}{p}$$

- **Proof.** (Non-admissibility in $\mathbf{K} \oplus \Box\bot$):
  - $\mathbf{K} \oplus \Box\bot$ is consistent because it is satisfied in the one point irreflexive frame to the right.
  - It follows in particular that a rule admissible in a logic L need not be admissible in its extensions.



Admissible Rules in Modal Logic

- The following rule is admissible in every normal modal logic.
- It is derivable in **GL** and **S4.1**, but it is not derivable in **K**, **D**, **K4**, **S4**, **S5**.

$$(\Diamond) \qquad \frac{\Diamond p \wedge \Diamond \neg p}{\bot}$$

- Löb's rule **(LR)** is admissible (but not derivable) in the basic modal logic **K**.
- It is derivable in **GL**. However, **(LR)** is not admissible in **K4**.

$$\textbf{(LR)} \qquad \frac{\Box p \to p}{p}$$

Admissible Rules in IPC

- The following rule is admissible in **IPC**, but not derivable:
  - Kreisel-Putnam rule (or Harrop's rule (1960), or independence of premise rule).

$$(KPR) \qquad \frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)}$$

  - (KPR) is admissible in **IPC** (indeed in any superintuitionistic logic), but the formula:

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

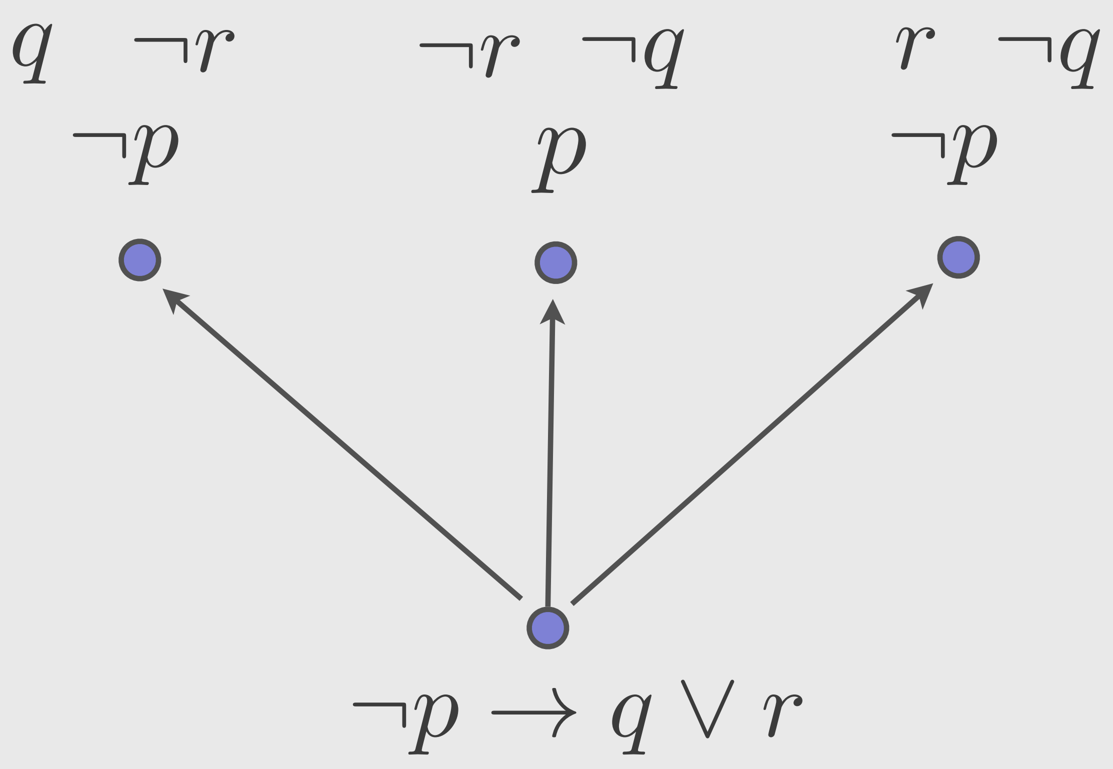
  - is not an intuitionistic tautology, therefore (KPR) is not derivable, and **IPC** is not structurally complete.
  - Note: **IPC** has a standard ***Deduction Theorem*** (only intuitionistically valid axioms are used in the classical proof)

(KPR) is Not Derivable: Proof

- Harrop's rule is derivable in **IPC** if the following is a tautology:

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

- The following Kripke model for **IPC** gives a counterexample:



Decidability of Admissibility

- Is admissibility **decidable**? I.e. is there an algorithm for recognizing admissibility of rules? (Friedman 1975)
- Yes, for many modal logics, as Rybakov 1997 and others showed.
- It is typically coNExpTime-complete (Jeřábek 2007).
- Decidability of admissibility is a major open problem for modal logic **K**.
- Recent results by Wolter and Zakharyaschev (2008) show e.g. the undecidability of admissibility for modal logic **K** extended with the universal modality.

Some Notes on Bases

- Is admissibility decidable for **IPC**? Rybakov gave a first postive answer in 1984. He also showed:
  - admissible rules do not have a finite basis;
  - gave a semantic criterion for admissibility.
- Admissibility in intuitionistic logic can also be reduced to admissibility in **Grz** using the Gödel-translation.
- Iemhoff 2001: there exists a recursively enumerable set of rules as a basis.
- Without proof, we mention that the rule below gives a singleton basis for the modal logic **S5**.

$$(\Diamond) \qquad \frac{\Diamond p \wedge \Diamond \neg p}{\bot}$$

Summary

- We have introduced the modal logic **K** and the intuitionistic calculus **IPC**.
- Have shown how they can be characterised by certain classes of Kripke frames.
- Discussed several proof systems for these logics.
- Introduced translations between logics and discussed how these can be used to transfer various properties of logics.
- Discussed the difference between admissible and derivable rules in modal, intuitionistic, and classical logic.

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