From syllogism to common sense: a tour through the logical landscape

Propositional logic 3

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Semantic equivalence and normal forms

- $\alpha \equiv \beta$ if $w\alpha = w\beta$ for all w
- Replacement Theorem: if $\alpha \equiv \alpha'$ then $\varphi[\alpha'/\alpha] \equiv \varphi$, i.e., if a subformula α of φ is replaced by $\alpha' \equiv \alpha$ in φ, then the resulting formula is equivalent to φ
- Negation normal form (NNF) is established by "pulling negation inwards", interchanging $\wedge$ and $\vee$
- Disjunctive normal form (DNF) of fct f is established by describing all lines with function value 1 in truth table
- Conjunctive normal form (CNF): analogous, dual

Functional completeness and duality

- Signature S is functional complete: every Boolean fct is represented by some fma in S
  - Examples: $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\rightarrow, 0\}$, $\{\uparrow\}$, $\{\downarrow\}$
  - Counterexample: $\{\rightarrow, \wedge, \vee\}$
- Dual formula: interchange $\wedge$ and $\vee$
- Dual function: negate arguments and function value
- Duality theorem: If α represents f, then α^δ represents f^δ.

Tautologies etc.

- α is a tautology if $w \models \alpha$ ($w\alpha = 1$) for *all* w
- α is satisfiable if $w \models \alpha$ for *some* w
- α is a contradiction if α is not satisfiable
- Satisfiability can be decided in nondeterministic polynomial time (NP) and is NP-hard.
- Analogous for tautology property: coNP-complete
- α is a logical consequence of X ($X \models \alpha$) if $\forall w(w \models X \Rightarrow w \models \alpha)$
- $\models$ enjoys certain general properties

And now ...

1. What happened last time?
2. A calculus of natural deduction
3. Hilbert calculi

What's in this section?

We want to ...

- find a means to "compute" $\models$ *syntactically*:
- define a derivability relation $\vdash$ by means of a calculus that operates solely on the structure of formulas
- prove that $\vdash$ and $\models$ are identical

The $\vdash$ calculus is of the Gentzen type

(Gerhard Gentzen, 1909–1945, German mathematician/logician, GÖ, Prague)

Basic notation

- Again, use α for formulas and X for sets thereof
- Write $X \vdash \alpha$ to denote: "α is derivable from X"
- Gentzen called the pairs (X, α) in the $\vdash$-relation sequents
- sequent calculus consists of 6 basic rules (for $\{\wedge, \neg\}$) of the form

$$\frac{\text{premise}}{\text{conclusion}}$$

The basic rules

$$\text{(IS)}\ \frac{}{\alpha \vdash \alpha}\ \text{(initial sequent)} \qquad \text{(MR)}\ \frac{X \vdash \alpha}{X' \vdash \alpha}\ (X' \supseteq X),$$

$$(\wedge 1)\ \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \qquad (\wedge 2)\ \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$$

$$(\neg 1)\ \frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} \qquad (\neg 2)\ \frac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta}$$

From W. Rautenberg: A Concise Introduction to Mathematical Logic, Springer, 2010.

- use convenience notation as for $\models$, see Slide 48 (last week)
- (IS) has no premises; initial sequences start derivations
- (MR): monotonicity rule
- $(\wedge 1)$, $(\neg 1)$, $(\neg 2)$ have two premises; $(\wedge 2)$ has two conclusions $\rightsquigarrow$ is actually 2 rules

Using the calculus

- Derivation = finite sequence $S_0, \dots, S_n$ of sequents where every S_i is either
  - an initial sequent or
  - is obtained by applying some basic rule to elements from $S_0, \dots, S_{i-1}$
- α is derivable from X, written $X \vdash \alpha$, if there is a derivation with $S_n = X \vdash \alpha$.

Examples

1	α	$\vdash$	α	(IS)	
2	α, β	$\vdash$	α	(MR) 1	
3	β	$\vdash$	β	(IS)	
4	α, β	$\vdash$	β	(MR) 3	
5	α, β	$\vdash$	$\alpha \wedge \beta$	($\wedge$1) 2, 4	$\Rightarrow$ $\{\alpha, \beta\} \vdash \alpha \wedge \beta$

1	$p \wedge \neg p$	$\vdash$	$p \wedge \neg p$	(IS)	
2	$p \wedge \neg p$	$\vdash$	p	($\wedge$2) 1	
3	$p \wedge \neg p$	$\vdash$	$\neg p$	($\wedge$2) 1	
4	$p \wedge \neg p$	$\vdash$	$\neg(p \wedge \neg p)$	($\neg$1) 2, 3	
5	$\neg(p \wedge \neg p)$	$\vdash$	$\neg(p \wedge \neg p)$	(IS)	
6	$\emptyset$	$\vdash$	$\neg(p \wedge \neg p)$	($\neg$2) 4, 5	$\Rightarrow$ $\vdash \top$

Derivable rules

- Derivations can be long (see exercise sheet)
- Use derivable rules as "shortcuts" for frequently occurring patterns in derivations
- Examples:

$$\frac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$$

¬-elimination

1	α	$\vdash$	α	(IS)
2	X, α	$\vdash$	α	(MR) 1
3	$X, \neg\alpha$	$\vdash$	α	supposition
4	X	$\vdash$	α	($\neg$2)

$$\frac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$$

reductio ad absurdum

1	$X, \neg\alpha$	$\vdash$	$\beta, \neg\beta$	supposition
2	$X, \neg\alpha$	$\vdash$	α	($\neg$1)
3	X	$\vdash$	α	¬-*elimination* 2

Further derivable rules

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$ cut rule

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$ $\to$-elimination

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$ $\to$-introduction — syntactic deduction theorem

$$\frac{X \vdash \alpha, \alpha \to \beta}{X \vdash \beta}$$ detachment rule — syntactic *modus ponens*

Relation between $\vdash$ and $\models$

- Goal: show $\vdash = \models$, i.e., $X \vdash \alpha$ iff $X \models \alpha$ for all X, α
- Direction $\subseteq$ or $\Rightarrow$: (semantical) soundness of $\vdash$
  (each fma derivable from X is a semantic consequence of X)
- Direction $\supseteq$ or $\Leftarrow$: (semantical) completeness of $\vdash$
  (each semantic consequence of X can be derived from X)

Soundness is the easier direction ...

Theorem (Soundness of $\vdash$)

$\vdash$ is semantically sound, i.e., $\forall X, \alpha : X \vdash \alpha \;\Rightarrow\; X \models \alpha$

Proof.

Let $X \vdash \alpha$. $\Rightarrow \exists$ valid derivation $S_1, \ldots, S_n$ with $S_n = X \vdash \alpha$.

Induction on n.

- $n = 1$. $\Rightarrow S_1 = \alpha \vdash \alpha$, and $\alpha \models \alpha$ obviously holds.
- $n \rightsquigarrow n+1$. Consider S_{n+1} in $S_1, \ldots, S_{n+1}$.
  - Either $S_{n+1} = \alpha \vdash \alpha$ (then argue as for $n = 1$)
  - or S_{n+1} is obtained by applying some rule, e.g., $\dfrac{S_i = X' \vdash \alpha'}{S_{n+1} = X \vdash \alpha}$
    - induction hypothesis: $X' \models \alpha'$
    - since rules preserve the consequence relation (see exercise), we can conclude $X \models \alpha$ $\square$

Finiteness

Another property that can be proven using induction on derivation length:

Theorem (Finiteness theorem for $\vdash$)

If $X \vdash \alpha$, then there is a finite subset $X_0 \subseteq X$ with $X_0 \vdash \alpha$.

Intuitive justification:

Every derivation has finite length

$\Rightarrow$ Only finitely many formulas can "accumulate" in X during a derivation

Formal consistency

- ... is a property crucial to the completeness proof
- ... will turn out to be the $\vdash$-equivalent of satisfiability

Definition:

- Set X of fmas is inconsistent if $X \vdash \alpha$ for all fmas α, consistent otherwise.
- X is maximally consistent if X is consistent but each $Y \supsetneq X$ is inconsistent

Observations:

- X inconsistent iff $X \vdash \bot$
  (for "$\Leftarrow$" use $\bot = (p \wedge \neg p)$ and rules $(\wedge 2)$, $(\neg 1)$)

$\rightsquigarrow$ X maximally consistent iff $\forall \alpha$: either $\alpha \in X$ or $\neg\alpha \in X$

Helpful properties of $\vdash$

Lemma

The derivability relation $\vdash$ has the following properties.

$\mathtt{C}^+$: $X \vdash \alpha$ iff $X, \neg\alpha \vdash \bot$ $\qquad$ $\mathtt{C}^-$: $X \vdash \neg\alpha$ iff $X, \alpha \vdash \bot$

Proof: Exercise.

This lemma helps with our goal of showing $\models \subseteq \vdash$:

- "$\models \subseteq \vdash$" iff $\forall X, \alpha : X \nvdash \alpha \Rightarrow X \not\models \alpha$
- By $\mathtt{C}^+$, $X \nvdash \alpha$ iff $X' := X \cup \{\neg\alpha\}$ is consistent
- By definition of $\models$, $X \not\models \alpha$ iff X' satisfiable

$\Rightarrow$ Suffices to show: consistent sets are satisfiable

Consistent sets are satisfiable (I)

Lemma (Lindenbaum's lemma)

Every consistent set $X \subseteq \mathcal{F}$
can be extended to a maximally consistent set $X' \supseteq X$.

(Adolf Lindenbaum, 1904–1941, Polish logician/mathematician, Warsaw)

Proof sketch:

- Enumerate all formulas $\alpha_0, \alpha_1, \ldots$
- For every $i = 0, 1, \ldots$:
  if $X \cup \{\alpha_i\}$ is consistent, then add α_i to X.
- X' is the limit of this extension procedure

Consistent sets are satisfiable (II)

Lemma ($\neg$)

For every maximally consistent set $X \subseteq \mathcal{F}$ and every $\alpha \in \mathcal{F}$:

$$X \vdash \neg\alpha \text{ iff } X \nvdash \alpha \qquad (\neg)$$

Proof.

"$\Rightarrow$" Due to consistency of X.

"$\Leftarrow$" If $X \nvdash \alpha$, then $X, \neg\alpha$ is consistent due to $\mathtt{C}^+$.
Since X is max. consistent, this implies $\neg\alpha \in X$.
Hence $X \vdash \neg\alpha$.

□

Consistent sets are satisfiable (III)

Lemma

Every maximally consistent set X is satisfiable.

Proof. Define valuation w by: $w \models p$ iff $X \vdash p$
Show by induction $\forall\alpha$: $X \vdash \alpha$ iff $w \models \alpha$.
(This implies $w \models X$, which completes the proof.)

- Base case ($\alpha = p$) follows from definition of w.
- Induction step for $\wedge, \neg$:

$$\begin{aligned}
X \vdash \alpha \wedge \beta &\Leftrightarrow X \vdash \alpha, \beta && \text{(rules } (\wedge 1), (\wedge 2))\\
&\Leftrightarrow w \models \alpha, \beta && \text{(induction hypothesis)}\\
&\Leftrightarrow w \models \alpha \wedge \beta && \text{(definition } \models)
\end{aligned}$$

$$\begin{aligned}
X \vdash \neg\alpha &\Leftrightarrow X \nvdash \alpha && \text{(lemma } (\neg))\\
&\Leftrightarrow w \not\models \alpha && \text{(induction hypothesis)}\\
&\Leftrightarrow w \models \neg\alpha && \text{(definition } \models)
\end{aligned}$$

□

Completeness!

Theorem (Completeness of $\vdash$)

$\vdash$ is semantically complete, i.e., $\forall X, \alpha : X \models \alpha \;\Rightarrow\; X \vdash \alpha$

Proof. Via contraposition.

- Assume $X \nvdash \alpha$.
- Then $X, \neg\alpha$ is consistent.
- Due to Lindenbaum's lemma:
  there is maximally consistent extension Y of $X, \neg\alpha$.
- Due to previous lemma: Y satisfiable
- Hence $X, \neg\alpha$ satisfiable.
- Therefore $X \not\models \alpha$.

$\square$

Interesting consequences of soundness + completeness

Theorem (Finiteness theorem for $\models$)

If $X \models \alpha$, then there is a finite subset $X_0 \subseteq X$ with $X_0 \models \alpha$.

Follows directly from finiteness theorem for $\vdash$
and soundness + completeness of $\vdash$.

Theorem (Propositional compactness theorem)

$X \subseteq \mathcal{F}$ is satisfiable iff each finite subset of X is satisfiable.

Follows directly from finiteness theorem for $\vdash$ with the observation that X unsatisfiable iff $X \models \bot$.

And so, Arthur and Bedevere and Sir Robin set out on their search to find the enchanter of whom the old man had spoken in scene twenty-four. Beyond the forest, they met Launcelot and Galahad, and there was much rejoicing. In the frozen land of Nador, they were forced to eat Robin's minstrels.

And there was much rejoicing.

A year passed. Winter changed into Spring.

Spring changed into Summer.

Summer changed back into Winter, and Winter gave Spring and Summer a miss and went straight on into Autumn.

Until one day ...

(from "Monty Python and the Holy Grail", 1975)

And now ...

1. What happened last time?
2. A calculus of natural deduction
3. Hilbert calculi

Hilbert calculi . . .

- are very simple logical calculi
- are based on arbitrary choice of logical tautologies as *axioms*
- use rules of inference to prove other tautologies from the axioms
- lead to more intuitive proofs than sequent calculi

(David Hilbert, 1862–1943, German mathematician, Königsberg, GÖ)

A standard Hilbert calculus

- Logical signature: $\neg, \wedge$
  (use $\alpha \rightarrow \beta$ as abbreviation for $\neg(\alpha \wedge \neg\beta)$)
- Set Λ of axioms: (5 schemes $\widehat{=}$ infinitely many axioms)

$$\begin{array}{ll}
\Lambda1 & (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \\
\Lambda2 & \alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta) \\
\Lambda3 & (\alpha \wedge \beta) \rightarrow \alpha \\
 & (\alpha \wedge \beta) \rightarrow \beta \\
\Lambda4 & (\alpha \rightarrow \neg\beta) \rightarrow \beta \rightarrow \neg\alpha
\end{array}$$

- Only one inference rule! Modus ponens:

$$\text{MP} \quad \frac{X \vdash \alpha, \ \alpha \rightarrow \beta}{X \vdash \beta}$$

(whenever α and $\alpha \rightarrow \beta$ are provable from X, then so is β)

Using the calculus

- Proof from X = finite sequence $\varphi_0, \ldots, \varphi_n$ of formulas where every φ_i is either
  - from $X \cup \Lambda$ or
  - is obtained by applying MP to two elements from $\varphi_0, \ldots, \varphi_{i-1}$
- α is provable from X, written $X \vdash \alpha$, if there is a proof from X with $\varphi_n = \alpha$.

Example

Proof of $X = \{p, q\} \vdash p \wedge q$

1	p	X
2	q	X
3	$p \rightarrow (q \rightarrow p \wedge q)$	$\Lambda2$
4	$q \rightarrow p \wedge q$	MP 1, 3
5	$p \wedge q$	MP 2, 4

Proof of $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$

1	$\beta \wedge \neg\alpha \rightarrow \neg\alpha$	$\Lambda3$
2	$(\beta \wedge \neg\alpha \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \neg(\beta \wedge \neg\alpha))$	$\Lambda4$
3	$\alpha \rightarrow \underbrace{\neg(\beta \wedge \neg\alpha)}_{\beta \rightarrow \alpha}$	MP 1, 2

Soundness

Theorem (Soundness of $\vdash$)

$\vdash$ is semantically sound, i.e., $\forall X, \alpha : X \vdash \alpha \;\Rightarrow\; X \models \alpha$

This is immediate to see:

- All axioms in Λ are tautologies (use truth tables).
- MP *preserves* tautologies, i.e.:
  if α and $\alpha \to \beta$ are tautologies, then so is β.

Hence every formula generated in a proof is a tautology.

Completeness

Theorem (Completeness of $\vdash$)

$\vdash$ is semantically complete, i.e., $\forall X, \alpha : X \models \alpha \;\Rightarrow\; X \vdash \alpha$

Proof uses the completeness of $\vdash$:

- $\vdash$ satisfies all basic rules of $\vdash$
  e.g., $(\wedge 2)\ \dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$ also holds for $\vdash$: $\dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$
  (to see this, use $\Lambda 3$ and MP)
- Therefore, $\vdash \subseteq \vdash$
- Since $\vdash = \models$, we obtain $\models \subseteq \vdash$

Hence every formula generated in a proof is a tautology.

Summary and outlook

- Prop. logic (PL) relies on the principles of
  bivalence and extensionality.
- PL formulas represent exactly the Boolean functions.
- Logical validity and consequence are defined
  via the $\models$ relation, based on valuations.
- Natural deduction (Gentzen-type sequent calculi) and
  Hilbert calculi both calculate the $\models$ relation syntactically.
- We haven't captured other types of calculi,
  such as tableau calculi or the resolution calculus.

Literature

Contents is taken from Chapter 1 of

W. Rautenberg:
A Concise Introduction to Mathematical Logic, Springer, 2010.

- This issue at Universitext: DOI 10.1007/978-1-4419-1221-3_1
- German version of 2008: DOI 10.1007/978-3-8348-9530-1

- PDF of Chapter 1 available in StudIP under "Dateien"

Thank you.