Dr. Mehul Bhatt, Dr. Oliver Kutz, Dr. Thomas Schneider WS 2011/12

From syllogism to common sense ...

Exercise Sheet 9: Quantification

To be discussed on 2 February 2012

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1. Consider the following ontology $\mathcal{O}$ that uses the concept names Animal, Marsupial, Koala, Lazy, Person, KoalaWithPhD, PhD, EucalyptForest, Australia, America and the role names hasHabitat, hasDegree.

   Show that $\mathcal{O}$ entails that neither Koala nor KoalaWithPhD can have any instances, that is, every interpretation that satisfies all axioms in $\mathcal{O}$ must interpret Koala and KoalaWithPhD as the empty set.

$$
\begin{array}{lcl}
\mathsf{Animal} & \sqsubseteq & \exists\,\mathsf{hasHabitat}.\top \\
\mathsf{Marsupial} & \sqsubseteq & \mathsf{Animal} \sqcap \forall\,\mathsf{hasHabitat}.(\mathsf{Australia} \sqcup \mathsf{America}) \\
\mathsf{Koala} & \equiv & \mathsf{Marsupial} \sqcap \exists\,\mathsf{hasHabitat}.\mathsf{EucalyptForest} \\
\mathsf{Koala} & \sqsubseteq & \mathsf{Lazy} \\
\mathsf{KoalaWithPhD} & \equiv & \mathsf{Koala} \sqcap \exists\,\mathsf{hasDegree}.\mathsf{PhD}
\end{array}
$$

$$\exists\,\mathsf{hasDegree}.\top \ \sqsubseteq\ \mathsf{Person}$$
$$\mathsf{Marsupial} \sqcap \mathsf{Person} \ \sqsubseteq\ \bot$$
$$\mathsf{Lazy} \ \sqsubseteq\ \mathsf{Person}$$

2. Consider the simplest Kripke semantics with constant domains for MPL (Modal Predicate Logic) given at the bottom of Slide 26 and at the following page of this exercise sheet. Show that the Barcan Formula BF on Slide 31 and its converse CBF on Slide 32 are valid under this semantics.

3. Consider the Kripke semantics with varying domains for MPL given at the following pages of this exercise sheet. Show that BF and CBF are not valid under this semantics. Under which restriction does each of them become valid?

See also the hints on the next pages.

Hints for exercises

2. We consider MPL with unary predicates only. Under the Kripke semantics with constant domains, models are tuples $M = (W, R, D, \mathrm{val})$ where

- (W, R) is a standard Kripke frame, where every world w is associated with an interpretation J_w that maps every unary predicate symbol P to a subset $J_w(P)$ of D;
- D *(domain)* is a nonempty set of possible individuals;
- $\mathrm{val} : V \to D$ *(valuation function)* is a function that assigns to every quantifiable variable an element from D.

The following conditions say when formulas are true at worlds. Let $M = (W, R, D, \mathrm{val})$ be a model and $w \in W$ a world.

$$
\begin{array}{lcl}
M, w \models P(x) & \Leftrightarrow & \mathrm{val}(x) \in J_w(P) \\
M, w \models \neg\varphi & \Leftrightarrow & M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \Leftrightarrow & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \varphi \vee \psi & \Leftrightarrow & M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models \Box\varphi & \Leftrightarrow & \text{for all } v \in W \text{ with } wRv : M, v \models \varphi \\
M, w \models \Diamond\varphi & \Leftrightarrow & \text{there is } v \in W \text{ with } wRv \text{ and } M, v \models \varphi \\
M, w \models \forall x.\varphi & \Leftrightarrow & \text{for all } d \in D : M(d/x), w \models \varphi \\
M, w \models \exists x.\varphi & \Leftrightarrow & \text{there is } d \in D : M(d/x), w \models \varphi
\end{array}
$$

The expression $M(d/x)$ denotes the model M with the valuation val changed to val′, where

$$
\mathrm{val}'(y) = \begin{cases} d & \text{if } y = x, \\ \mathrm{val}(y) & \text{otherwise,} \end{cases}
$$

i.e., val′ maps x to d and treats all other variables as val does.

The Barcan formulas are

$$
\begin{array}{ll}
\text{(BF)} & \forall x.\Box P(x) \to \Box\forall x.P(x) \\
\text{(CBF)} & \Box\forall x.P(x) \to \forall x.\Box P(x)
\end{array}
$$

3. Under the Kripke semantics with varying domains, models are tuples $M = (W, R, \mathrm{val})$ where

- (W, R) is a standard Kripke frame, where every world w is associated with

  - its own *domain* D_w
  - and an interpretation J_w that maps every unary predicate symbol P to a subset $J_w(P)$ of D_w;
- val : $V \to \bigcup_{w \in W} D_w$ a *valuation*.

The above truth conditions can be reused, except for the quantifier cases, which are modified as follows. Let $M = (W, R, \mathrm{val})$ be a model and $w \in W$ a world.

$$M, w \models \forall x.\varphi \;\Leftrightarrow\; \text{for all } d \in D_w : M(d/x), w \models \varphi$$
$$M, w \models \exists x.\varphi \;\Leftrightarrow\; \text{there is } d \in D_w : M(d/x), w \models \varphi$$