# Course 16
# Geometric Data Structures for Computer Graphics

## Voronoi Diagrams

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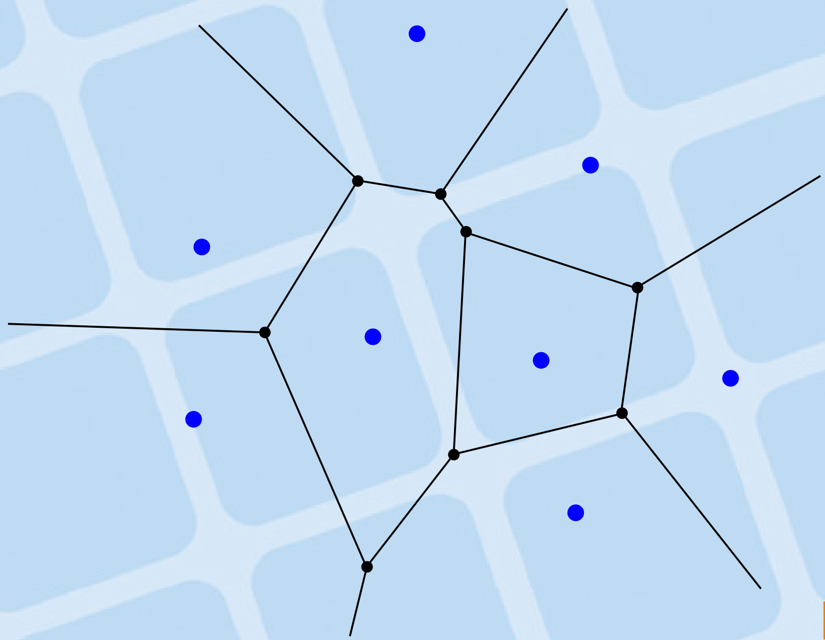
# Definition Voronoi Diagram

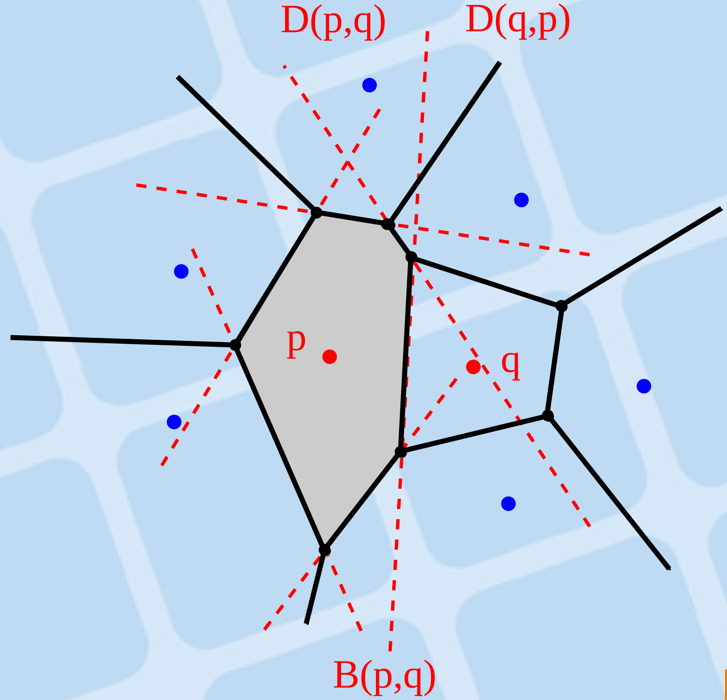
Classical Voronoi Diagram in 2-D



- Set of sites $S$ in the Euclidean plane
- Subdivision into regions of the same neighborship
- Well-known concept Biology, Economics, CS, ...

Voro Glide

# Abstract definition



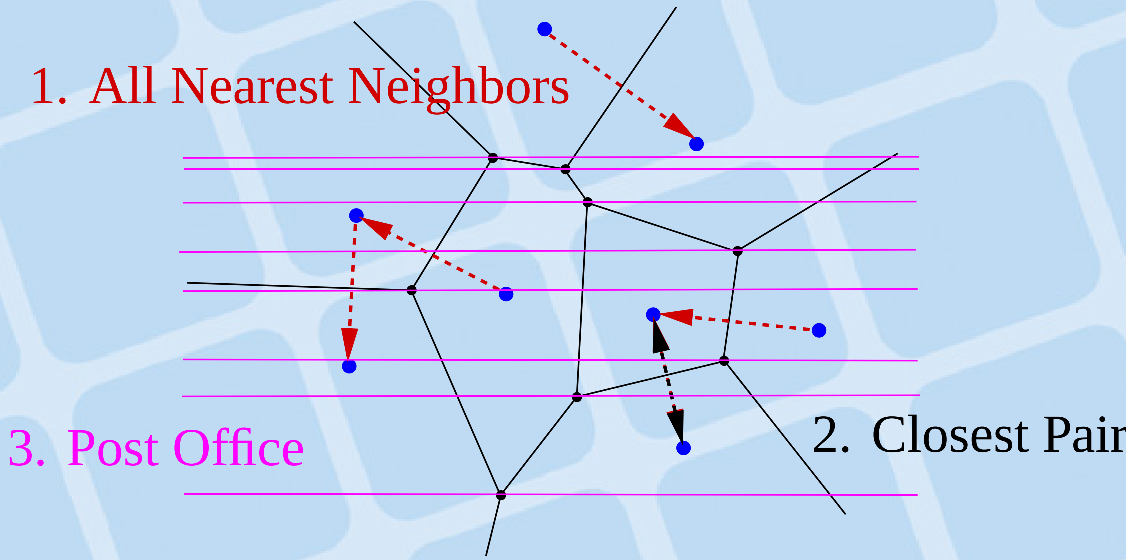

- *Bisector*: $B(p,q) = \{x \mid d(p,x) = d(q,x)\}$
- *Halfplane*: $D(p,q) = \{x \mid d(p,x) < d(q,x)\}$
- *Voronoi Region*: $\mathrm{VR}(p,S) = \bigcap_{q \in S, q \neq p} D(p,q)$
- *Voronoi Diagram*: $V(S) = \bigcup_{p,q \in S, p \neq q} \overline{\mathrm{VR}(p,S)} \cap \overline{\mathrm{VR}(q,S)}$

# Properties

- Voronoi Diagram
  - Graph
  - Complexity: $O(n)$ egdes and vertices,
    Region: $6$ boundary edges in the average
    (Application of Euler-Formula)
  - Data Structure: DCEL, Adjacency List
  - Simple linear structure, represents a decomposition of the plane in cells
  - Implementations: LEDA, CGAL, Qhull, ...

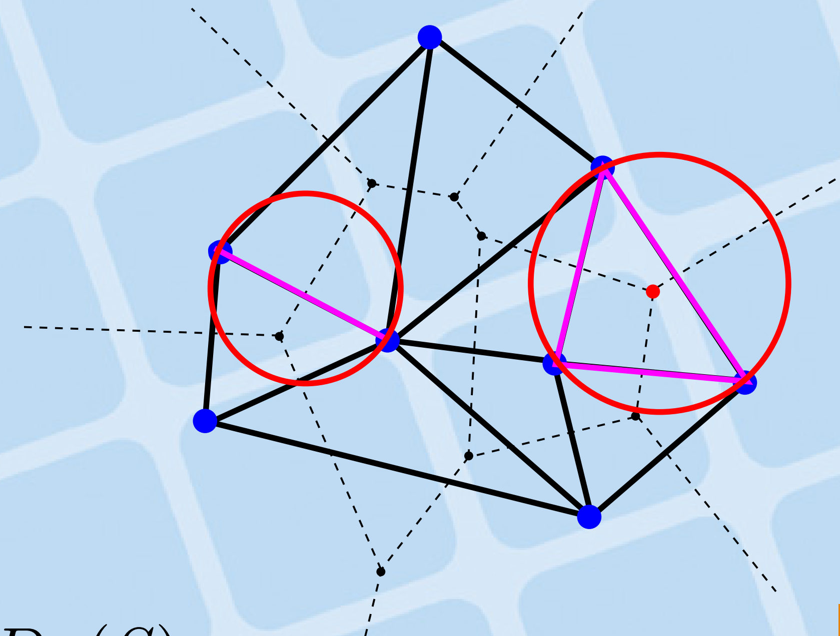
# Simple Applications

Voronoi Diagram of a set of points is given




- 1. All Nearest Neighbors: $O(n)$ time
- 2. Closest Pair: $O(n)$ time
- 3. Post Office Problem/Locus Approach: Query time: $O(\log n)$
  - Simple preprocessing: $O(n^2)$ time and space
    More complex: $O(n)$ (Edelsbrunner)

# Delaunay Triangulation: The Dual



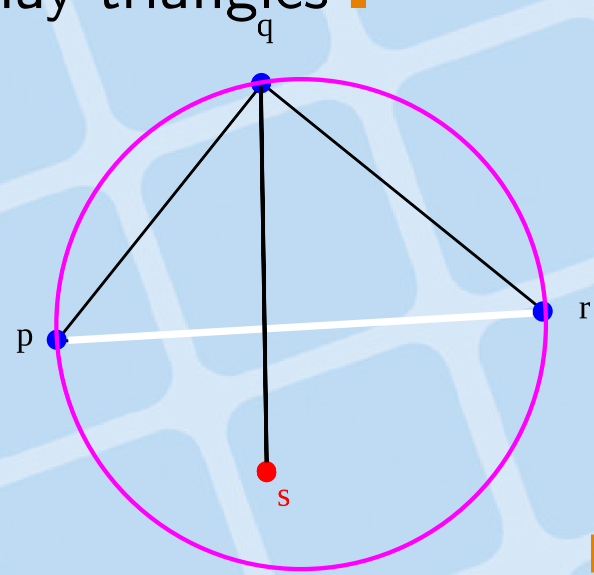
- The dual graph $D_T(S)$
- Triangulation of $S$, $(n-1)$ triangles
- Charaterizations
  - Triangle: Circumcircle contains no other site
  - Edge: Circle contains no other site
- Maximizes the minimum angle

# Computation

- Lower bound: $\Omega(n \log n)$
  - Reduction to the Convex Hull (Shamos)
  - Reduction to $\epsilon$-closeness (Zhu and Mirzaian)
- Construction: $O(n \log n)$
  - Incremental
  - Divide and Conquer
  - Sweep
  - Delaunay Triangulation

# Simple Incremental Construction

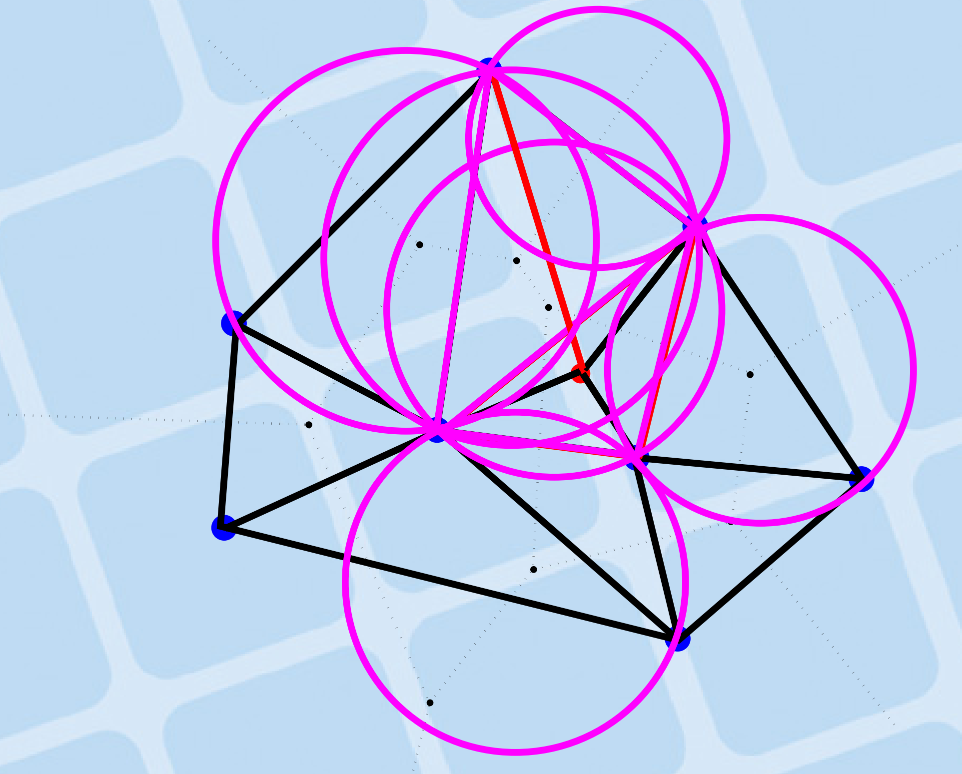
- Works on the Delaunay Triangulation
- Easy to implement/generalize
- Using *edge flips*
- Assume that $DT(\{p_1, p_2, \ldots, p_{i-1}\})$ was constructed
- Insert $p_i$
- Conflicts with Delaunay triangles



# Simple Incremental Construction

Insert $p_i$

- Determine triangle
- Sucessively remove conflicts by Edge-flips



# More Applications

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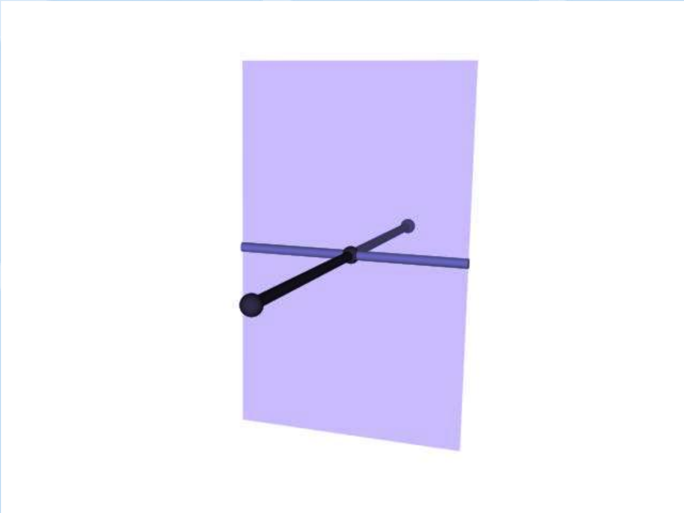
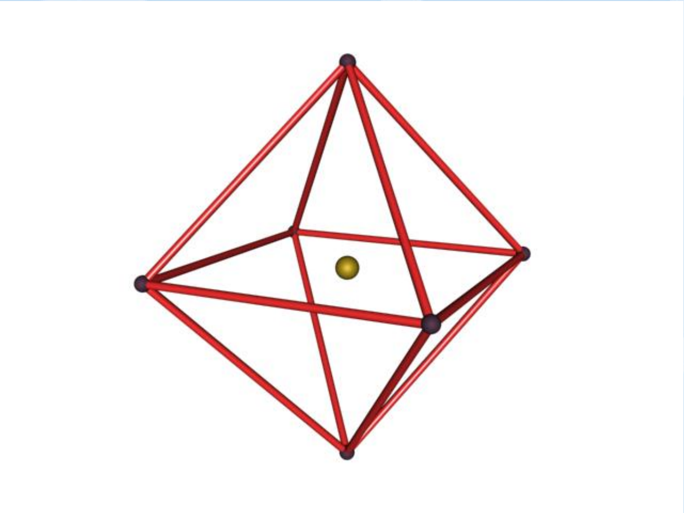
Assume that the diagram is given:

- $k$-th nearest neighbor of point $x \notin S$: $O(k \log^2 n)$ expected time
- Minimum Spanning Tree, TSP-Heuristic: $O(n \log n)$
- Largest empty circle in area $A$: $O(n)$
- Smallest enclosing circle/square: $O(n)$
- Localization problems (Hamacher)
- Clustering of objects (Dehne, Noltemeier)

# Voronoi Diagram in 3-D

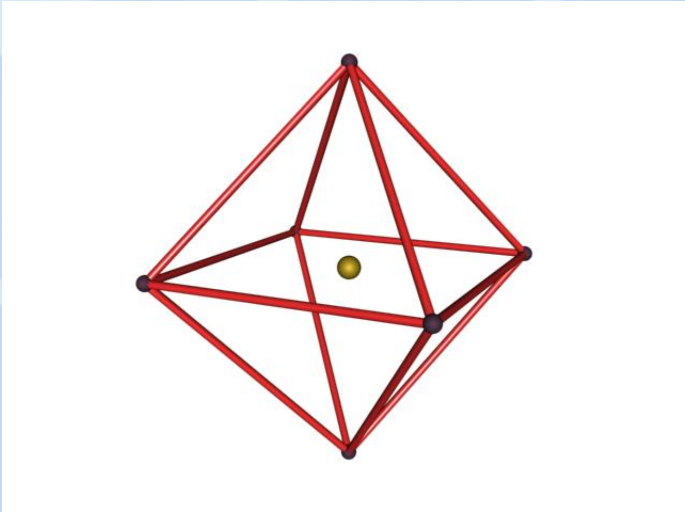


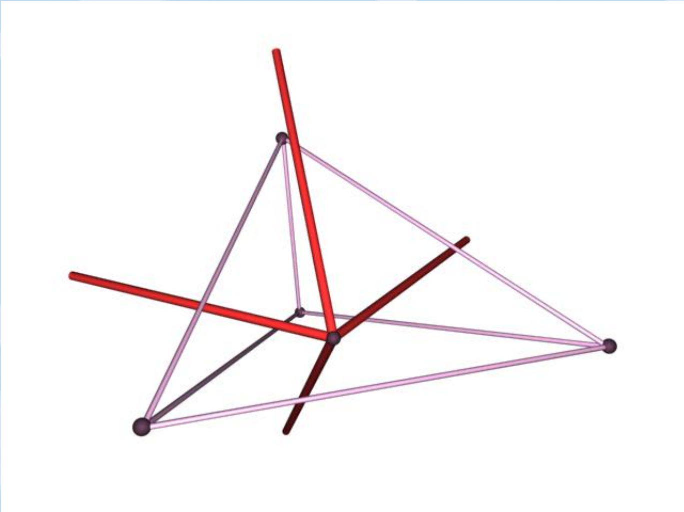


Set of points in the Euclidean 3D Space

- Bisector: Hyperplane
- Region: Intersection of halfspaces bounded by bisectors, 3D convex polyhedron
- Boundary of region: Facets, edges, vertices
- Decomposition of the space into 3D convex cells

# Voronoi Diagram in 3-D

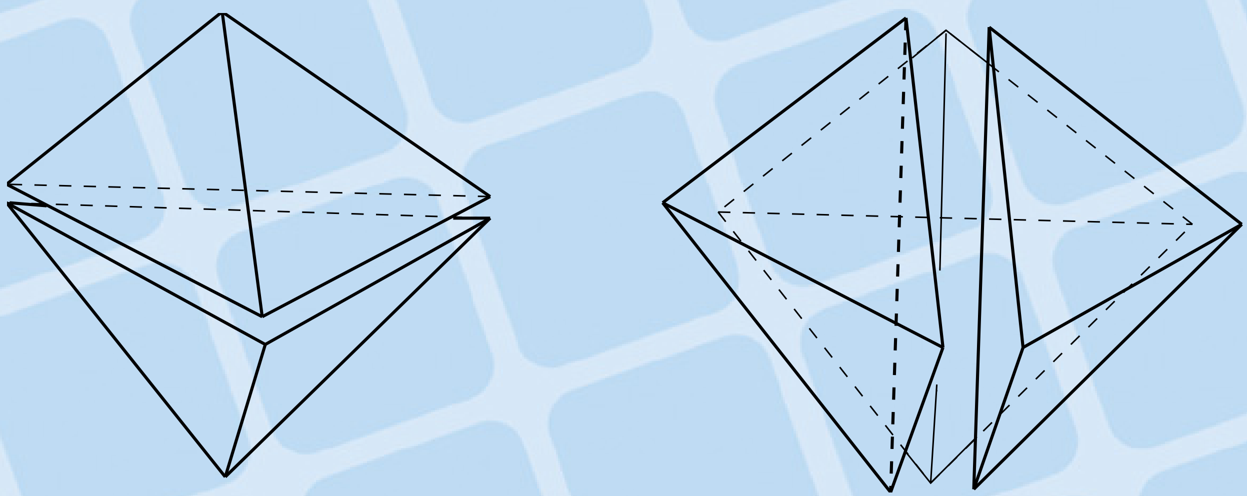




- Delaunay triangulation
  - Tetrahedron for every vertex
  - Triangle for every edge
  - Edge for every facet
  - Delaunay Tetrahedon: Circumsphere of four points is empty
- Unfortunately no demo software :-((

# Voronoi Diagram in 3-D

- Complexity:
  - $\Theta(n^2)$
  - Uniformly distributed: $O(n)$
- Construction:
  - Similar incremental approach: 3D edge flips in $\Theta(n^2)$



Two–into–three tetrahedra flip for five sites

# Voronoi Diagram in 3-D

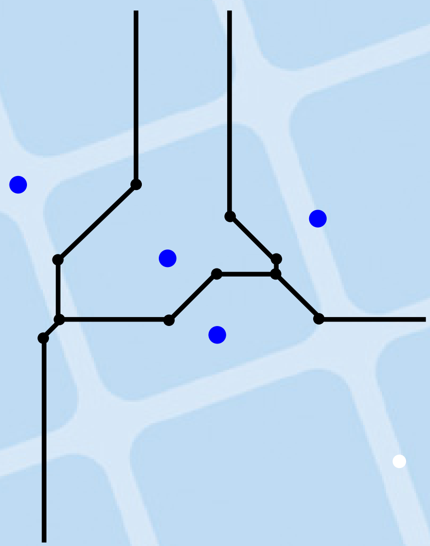
Application:

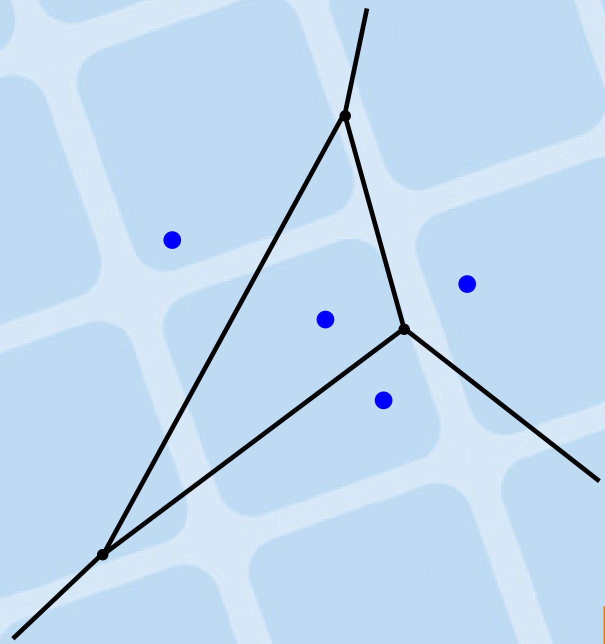
- Generalizations of 2-D applications
- For example: Post Office Problem, Smallest enclosing ball, All nearest neighbors, etc.

# Other generalizations

Other metrics

- $L_1$-Metric ($L_\infty$-Metric)
- Convex distance functions





More generalizations: weights, other objective (z.B. farthest points), colors, . . .